hope that the new organization might carry forward these aims with still greater effectiveness, coöperating, on the one hand, with the various well-organized secondary associations, and, on the other hand, with the American Mathematical Society in its chosen field of scientific research, but being careful to encroach upon neither of these fields.

The meeting was then resolved into a committee of the whole for the consideration, section by section, of a constitution and by-laws, tentative drafts of which had been prepared in advance. After three hours of patient and painstaking deliberation, all mooted questions were settled except the name of the new organization. This was left to a committee to choose from the eighteen different variations which had been proposed and to report the following morning. A committee was also delegated to assist the temporary secretary in smoothing out any verbal inconsistencies or inaccuracies in the constitution and by-laws. The name finally chosen by the committee was adopted without a dissenting vote, as embodying more favorable points and fewer objections than any other that had been suggested.

CONSTITUTION.

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad interim*.

Article IV—Meetings.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

Article V—Sections.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

The initiation fee shall be waived in case of those who join the Association before April 1, 1916, and this clause shall be dropped after its provisions have been fulfilled.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments.

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

Twelve members of the Council shall be elected at the first meeting of the Association, and the secretary shall draw lots to determine which four of those elected shall serve for one, for two, and for three years respectively. (This clause shall be dropped after its provisions have been fulfilled.)

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

This shall not be construed to affect existing contracts with any subscribers or news agencies for the year 1916, who may decline to readjust on the new basis. (This clause shall be dropped after its provisions have been fulfilled.)

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

As the provisions of the By-laws with respect to the nomination and election of officers could not be fulfilled in the first instance, a special resolution at the meeting on Thursday provided for a nominating committee, consisting of L. E. DICKSON, University of Chicago; D. R. CURTISS, Northwestern University; H. L. RIETZ, University of Illinois; S. E. RASOR, Ohio State University; and R. E. ROOT, United States Naval Academy. This committee reported immediately after the final adoption of the Constitution and By-laws on Friday morning. Their report, after full opportunity for further nominations from the floor, was adopted and the following officers were elected:

For President, E. R. HEDRICK, University of Missouri;
For Vice-Presidents, E. V. HUNTINGTON, Harvard University, and G. A. MILLER, University of Illinois;
For Secretary-Treasurer, W. D. CAIRNS, Oberlin College;
For additional members of the Executive Council:

To serve for one year.

D. N. LEHMER, University of California,
R. E. MORITZ, University of Washington,
K. D. SWARTZEL, Ohio State University,
OSWALD VEBLEN, Princeton University.

To serve for two years.

R. C. ARCHIBALD, Brown University,
FLORIAN CAJORI, Colorado College,
M. B. PORTER, University of Texas,
J. W. YOUNG, Dartmouth College.

To serve for three years.

B. F. FINKEL, Drury College,
E. H. MOORE, University of Chicago,
J. N. VAN DER VRIES, University of Kansas,
ALEXANDER ZIWET, University of Michigan.

The Council met on Friday afternoon and immediately completed its quota by the appointment of the Committee on Publications consisting of H. E. SLAUGHT, of the University of Chicago, managing editor, together with R. D. CARMICHAEL, University of Illinois, and W. H. BUSSEY, University of Minnesota.

The Council appointed a committee consisting of ALEXANDER ZIWET and K. D. SWARTZEL to act with power in negotiating with the owners of the AMERICAN MATHEMATICAL MONTHLY with a view to securing this as the official journal of the ASSOCIATION. The result of these negotiations was that the MONTHLY was formally transferred to the ASSOCIATION and thus, under these new auspices, it will begin its twenty-third year of continuous service.

The Council empowered the Committee on Publications to fill out the Editorial Board by the appointment of associate editors. This has been done as follows:

R. P. BAKER, University of Iowa; W. C. BRENKE, University of Nebraska; A. COHEN, Johns Hopkins University; B. F. FINKEL, Drury College; L. C. KARPINSKI, University of Michigan; G. H. LING, University of Saskatchewan; HELEN A. MERRILL, Wellesley College; U. G. MITCHELL, University of Kansas; W. H. ROEVER, Washington University; D. A. ROTHROCK, Indiana University; C. H. SLICHTER, University of Wisconsin; and D. E. SMITH, Columbia University.

The Editorial Board will be further organized by the appointment of committees in the various departments.

The Council also appointed a Standing Committee on Finance consisting of the President, the Secretary and the Managing Editor; also a special committee to secure a bond for the Secretary-Treasurer in the amount of $2,000.

The Secretary was instructed to draw lots to determine the terms of service of the twelve members of the Council just elected, the result being as indicated above.

The Council received formal applications from duly authorized representatives of three states requesting authority for organizing Sections of the ASSOCIATION; namely, from Kansas, Missouri and Ohio. The Kansas meeting was held early in the Autumn, the Missouri meeting at Thanksgiving time, and the Ohio meeting on Thursday afternoon at Columbus, the latter having some thirty-five delegates present. The Council appointed a committee consisting of E. R. HEDRICK, ALEXANDER ZIWET, and K. D. SWARTZEL, to formulate the terms under which such petitions may be granted, as provided by the Constitution, and to act with power on these and other similar petitions which may be received before the next meeting.

Finally, the Council authorized the President, Secretary, and Managing Editor to act with power in respect to all names proposed for membership during the formation period.

Although no program had been originally planned, aside from the arduous business of organizing the new ASSOCIATION, yet the temporary committee on arrangements was most fortunate in securing Professor L. C. KARPINSKI, of the University of Michigan, to give his illustrated address on "The Story of Algebra," which he did on Friday morning immediately following the business session. It is not too much to say that for the space of an hour he both charmed and edified an enthusiastic audience of approximately one hundred persons.

Hearty thanks are due to the local committee of arrangements, under the direction of Professor S. E. RASOR, for the accommodations provided for these meetings, including especially the luncheon on Friday given by the Department of Mathematics of Ohio State University to all mathematicians attending the Columbus meetings. Thanks are also due to the members of the AMERICAN MATHEMATICAL SOCIETY who were in attendance at Columbus for the very hearty cordiality and coöperation extended by them to the new ASSOCIATION.

For further notes on the Columbus meeting see the last pages of this issue.

W. D. CAIRNS, *Secretary.*

DIGITAL RECKONING AMONG THE ANCIENTS.

By LEON J. RICHARDSON, University of California.

Primitive man developed his notions of number to no small extent by aid of the fingers. Hence his habit of counting in systems of five and ten. As social relations grew, the needs of communication led to certain uniformities of practice. These signs and arithmetical processes must have differed among tribes; at the same time the fixed elements in the case, namely the ten fingers, made for similarities. Such devices for reckoning, moreover, once hit upon, especially if marked by superior convenience, tended to spread from tribe to tribe and from region to region, just as cleverly devised modes of writing numbers passed with little change from the Egyptians to the Phœnicians and thence to Palmyra and the Syrians.

I. Use of the Fingers in Representing Numbers.

The practice of indicating numbers on the fingers was common among ancient Egyptians, Babylonians, Greeks, and Romans. Moreover, the system was eventually developed to such an extent that all numbers from one to 10,000, and sometimes even larger numbers, could be so expressed. References to the subject are often met with in Greek and Latin literature, authors usually taking it for granted that their readers will understand details. Pliny the Elder (*Natural History*, 34, 7, 33) says that "king Numa dedicated a statue of two-faced Janus . . . the fingers being put in a position to show 365 . . . and thus to represent him as the god of time and duration." Macrobius (1, 9, 10), referring to copies of this statue, says the number 300 was "held" in the right hand and 65 in the left. A statue of the philosopher Chrysippus, who devoted much attention to mathematics, had "the fingers drawn together so as to indicate numbers" (Sidonius: *Epist.*, 9, 9, 14). The ring-finger, for example, when bent, indicated the number six (Macrobius, 7, 13, 10). The sign for 500 was made *flexo pollice* (Quintilian, 11, 3, 117). St. Jerome (*Adversus Iovinianum*, 1, 3) informs us that "Thirty is associated with marriage. For the very union of the fingers, as if embracing, . . . depict the husband and wife. Sixty indeed is associated with widows, for the reason that they are placed in straits and misery. . . . The number one hundred however (pray, reader, attend carefully) passes over from the left hand to the right . . . and forming a circle bodies forth the crown of virginity." Cassiodorus—historian, statesman and monk of the sixth century—in commenting on the sixtieth Psalm remarks: "The numbering of this Psalm moreover is not barren of interest. For the number sixty belongs fitly to celibates and widows, being represented by a bound position of the fingers."

The practise of representing numbers on the fingers gave rise to the English word *digit* as a name for each of the numerals below ten. The earliest known instance of this word in our literature, namely in the works of John de Trevisa under date of 1398, takes the form *digitus*, the Latin word having developed that meaning in the late middle ages, notably in books on algorism.

Fortunately we have not been left in the dark as to how the individual numbers were represented, for the complete system is set forth by the Venerable Bede and by Nicolaus Rhabda of Smyrna, both of the eighth century, men dwelling in widely separated parts of the world, one writing in Latin and the other in Greek. Bede's account, which antedates all others that have come down to us, is given in his book *De loquela per gestum digitorum* and Rhabda's is incorporated in the work of Nicolaus Caussinus *De eloquentia sacra et humana* (Paris, 1636). There exist still other accounts, notably one contained in the Persian and Arabic lexicon of Ghiyás and translated into English by E. H. Palmer. These writers set down what had been traditional, doubtless, from early times, for they agree substantially among themselves and their matter harmonizes in the main with the scattered and casual observations of classical authors.

From the foregoing sources we learn that numbers were expressed by the following signs. The hand, unless otherwise stated, was held upward, the palm flat, the fingers together, except the thumb, which did not touch the second finger.

1—5th finger of the left hand bent at the middle joint.
2—4th and 5th fingers bent at the middle joint.
3—3d, 4th and 5th fingers bent at the middle joint.
4—3d and 4th fingers bent at the middle joint.
5—3d finger bent at the middle joint.
6—4th finger bent at the middle joint.
7—5th finger closed on the palm.
8—4th and 5th fingers closed on the palm.
9—3d, 4th and 5th fingers closed on the palm.
10—Tip of the 2d (or index) finger touched the middle joint of the thumb.
11—The signs for 10 and 1 were made coincidentally. The same method applied to numbers from 12 to 19.
20—The thumb was placed between the 2d and 3d fingers in such a way that the thumb nail touched the middle joint of the 2d finger.
30—The thumb and the 2d finger formed a circle.
40—The thumb and 2d finger stood erect and close together.
50—The thumb, bent at both joints, rested on the palm.
60—The 2d finger was bent forward over the thumb, which remained in the position just described.
70—The first joint of the 2d finger rested upon the first joint of the thumb, which was held nearly straight.
80—The tip of the 2d finger rested upon the first joint of the thumb.
90—The thumb was bent over the first joint of the 2d finger.

The signs so far described were all made with the left hand. We now pass to the part played by the other hand. The sign for 100 did not differ from the sign for 10, except that it was made with the right hand. Similarly related were the signs for 200 and 20, 300 and 30 and so on through 900 and 90.

The sign for 1,000 did not differ from the sign for 1, except that it was made with the right hand. Similarly related were the signs for 2,000 and 2, 3,000 and 3 and so on through 9,000 and 9.

The sign for 10,000 was made by laying the left hand on the chest. 20,000, 30,000 and so on through 90,000 were made by touching various parts of the body

with the same hand. (See Bede, 692.) Similarly the signs for 100,000–900,000 were made by touching corresponding parts of the body with the right hand. The sign for 1,000,000 was the hands clasped, the fingers being interlocked.

II. Use of the Fingers in Counting and Reckoning.

Let us now turn to the other phase of the subject, the use of the fingers in counting and reckoning. The Homeric verb πεμπάξειν means "to count on the five fingers," "to count by fives." Herodotus (6, 63) employs the expression ἐπὶ δακτύλων συμβάλλεσθαι meaning "to reckon on the fingers." Significant also in this connection is a passage in Aristophanes (*Vespæ*, 655–657): "Hear then . . . and first of all do an easy sum—not with counters, but with your fingers—the tribute collectively which accrues to us from the cities."

There is a verse in Plautus (*Miles Gloriosus*, 204) which runs: "He reckons the pros and cons on the fingers of the right hand" (*dextera digitis rationem conputat*). To a Roman familiar with current mathematical usages the expression might convey something more than appears in our literal translation. The clue to the subtler meaning may be found in several ancient writings, but is also to be gathered, as it chances, from Sir Thomas Browne's *Pseudodoxia Epidemica* (iv, iv, 186): "On the left [hand] they accounted their digits and articulate numbers unto an hundred, on the right hand hundreds and thousands." Accordingly the "pros and cons" aforesaid, being reckoned on the right hand, are by implication many in number—jocosely represented as a hundred or more. A similar point is made by the poet Juvenal (*Saturnae*, x, 246): "If one has any faith in great Homer, [Nestor] was an instance of life inferior in duration only to the crow's. Happy was he indeed who put off the hour of his death so long and at last begins to count his years on his right hand" (*suos iam dextra conputat annos*).

An important phrase for our purpose is found in a letter that Cicero once wrote to his friend Atticus, the capitalist (*Ad Atticum*, 5, 21, 12–13): "Everybody present exclaimed that nothing was more shameless than Scaptius, who was not satisfied with 12 per cent., compound interest. . . . Lately a decree of the senate has been passed . . . on the subject of creditors fixing the rate at 12 per cent., simple interest. What difference this makes, if I know your skill at reckoning, you have certainly computed." The phrase "if I know your skill at reckoning" is literally "if I know your fingers" (*si tuos digitos novi*).

Ovid (*Fasti*, 3, 123) has the expression: "the fingers by the aid of which we are wont to count" (*digiti per quos numerare solemus*). But one of the most illuminating passages bearing on our subject is found in Quintilian (1, 10, 35): "As to geometry, people admit that attention to it is of advantage in tender years; for they allow that by this study the thinking powers are excited, the intellect sharpened and quickness of perception produced; but they fancy that it is not, like other sciences, profitable after it has been acquired, but only while it is being studied. Such is the common opinion respecting it. Not without reason, however, have the greatest men devoted much attention to this science;

for while geometry comprises numbers and forms, a knowledge of numbers assuredly is necessary not only to a speaker, but to any one taking even the first steps along the path of learning. For pleading cases in court it is very often in request. On these occasions, to say nothing of becoming confused about sums, if a speaker, by any uncertain or awkward movement of the fingers, differs from the accepted mode of calculation, he is thought to be poorly trained."

Pliny the Younger (2, 20, 3), speaking of a man occupied with thoughts about becoming the heir of a rich woman, says "He moves his lips, keeps his fingers going, reckons" (*movet labra, agitat digitos, computat*). Again Suetonius (*Claudius*, 21) makes this observation on the Roman emperor Claudius: "He gave many gladiatorial shows. . . . There was no form of entertainment at which he was more familiar and free, even thrusting out his left hand, like the commons, and counting aloud on his fingers the gold pieces which were paid to the victors" (*aureos . . . voce digitisque numeraret*).

Apuleius (*Apologia*, 89) protests that a certain lady's age has been represented as sixty, when she was really thirty: "If instead of ten you had said thirty years, you might seemingly have erred in the manual sign of the calculation, [in the former case, the tip of the forefinger touching the middle joint of the thumb,] in the latter those fingers forming a circle. Forty, however, is shown by the flat palm—the simplest of manual signs—and you increase that number by half. An error in the manual sign is out of the question, unless, supposing Pudentilla to be thirty, you counted both consuls with each year."

When we come down to the fourth and fifth centuries A. D., we find St. Jerome (*Adversus Iovinianum*, 1, 46) saying: "He shows that wives are wont to be selected more on the basis of wealth than of character and that many are guided, not by their eyes, but by their fingers, in marrying" (*multos non oculis sed digitis uxores ducere*). In the same period St. Augustine (*City of God*, 18, 53) writes: "He puts aside the fingers of the computers (*calculantium digitos*) and orders silence, who says 'It is not for you to know the times, which the Father hath put in His own power.'"

III. The Four Operations Represented by Use of the Fingers.

Enough has been presented to show how common among the ancients was the use of the fingers in dealing with numbers. The illustrations have been drawn for the most part from the Romans, to be sure, but similar material could be found among the Greeks and other peoples. It is worth remarking in this connection that no trace of finger computation of the sort we have described has been found among the Hindus, to whom ultimately we owe our own system of numerals. It is now time to consider how the fingers were employed in mathematical processes. This is not easy to answer in detail. The case of addition, however, is fairly well understood. When a series of numbers was to be added, the fingers at the outset were made to indicate the first number. The second number was then added mentally and the fingers put in a position to indicate the sum. The third number in its turn was added mentally and the fingers

changed to show the new sum. In this way a person proceeded until he arrived at the final sum. (See Marquardt, 1: 99.) The sight of the fingers was an aid in performing the successive steps of the addition. In subtraction the fingers seem to have been similarly used, according to evidence that has reached us. The same may be said of multiplication and division, for the former may be performed by a series of additions and the latter by a series of subtractions.

Concerning the matter of multiplication, some curious evidence has come to light. Dacia, which in ancient geography corresponds in the main to modern Roumania and Transylvania, became a Roman province under Trajan. It was in fact governed from Rome from about 101 to 256 A. D. During that period four military roads were constructed and several forts were built to protect the inhabitants from the incursions of surrounding barbarians. Numerous colonists from Italy settled in the country. The Dacians on their part adopted the religion and language of the conquerors. The Roman occupation, though comparatively short, may still be traced in Greek and Latin literature, in monoliths, inscriptions and coins, as well as in the language and customs of the Roumanians. Interestingly enough, the Walachian peasants, who dwell in southern Roumania, have preserved an old method of multiplication on the fingers. How old the custom is, or whence derived, no one can tell. It can hardly be of local origin, for it is not absolutely unique. It may hark back to oriental or Greek sources, for an algorism manuscript of about 1200 A. D., now preserved at Heidelberg, contains something similar. (See Cantor, 1: 780.) Again it may be of Roman origin. The last theory is supported by the fact that similar usages, probably of Roman origin, have been noted among French peasants. (See Cantor, 1: 447; also Sittl, p. 262.) Moreover, there were current in the middle ages similar mathematical usages which almost surely arose from Roman sources. The method of multiplication just mentioned is as follows:

TABLE I.

1st cycle $(6\cdots10)$ formula: $10(e+e')+cc'$.
2d cycle $(11\cdots15)$ formula: $15(e+e')+cc'+75$.
3d cycle $(16\cdots20)$ formula: $20(e+e')+cc'+200$.
4th cycle $(21\cdots25)$ formula: $25(e+e')+cc'+375$.
5th cycle $(26\cdots30)$ formula: $30(e+e')+cc'+600$.
. . . . . .
nth cycle $[5n+1\cdots5(n+1)]$ formula: $5(n+1)(e+e')+cc'+5^2(n^2-1)$.

ABBREVIATIONS:

e = extended fingers of the right hand.
e' = extended fingers of the left hand.
c = closed fingers of the right hand.
c' = closed fingers of the left hand.

To illustrate the foregoing formulæ, take a problem falling within the first cycle. How many are 7×7? Hold up each hand clenched. Extend the right thumb and index finger (= 7). Extend the left thumb and index finger (= 7). [Note that in this cycle the fingers beginning with the thumb have values respectively of 6, 7, 8, 9, 10.] How many fingers are extended? [The thumbs count as fingers.] Four. $4 \times 10 = 40$. How many fingers are closed? Three on the

right hand and three on the left. $3 \times 3 = 9$. Therefore $7 \times 7 = 49$. This process, it will be observed, presupposes a knowledge of the multiplication table in the modern sense through the fives.

Again, how many are 6×8? Hold up each hand clenched. Extend the right thumb (= 6). Extend the left thumb, index finger and middle finger (= 8). How many fingers are extended? Four. $4 \times 10 = 40$. How many fingers are closed? Four on the right hand and two on the left. $4 \times 2 = 8$. Therefore $6 \times 8 = 48$.

Second cycle problem: How many are 12×13? Hold up each hand clenched. Extend the right thumb and index finger (= 12). Extend the left thumb, index finger and middle finger (= 13). [In this cycle the fingers have the values of 11, 12, 13, 14, 15.] How many fingers are extended? Five. $5 \times 15 = 75$. How many are closed? Three on the right hand and two on the left. $3 \times 2 = 6$. $75 + 6 + 75 = 156$. Therefore $12 \times 13 = 156$.

In order to multiply two numbers belonging to different cycles, *e. g.*, 9×13, the larger number is so divided as to bring the problem within a single cycle. *E. g.*, $9 \times 13 = (9 \times 6) + (9 \times 7)$.

In actual practice the Walachians do not go beyond the first cycle. The other cycles as here set forth have been made out inferentially.

The formulæ given above may assume a slightly different form:

TABLE II.

1st cycle (6···10) formula: $10(e + e') + cc'$.
2d cycle (11···15) formula: $10(e + e') + ee' + 100$.
3d cycle (16···20) formula: $20(e + e') + cc' + 200$.
4th cycle (21···25) formula: $20(e + e') + ee' + 400$.
5th cycle (26···30) formula: $30(e + e') + cc' + 600$.

As regards the nth cycle: when n is an odd number, the general formula is the same as that given in Table I; when, however, n is an even number, the general formula is:

$$n\text{th cycle } [5n + 1 \cdots 5(n + 1)] \text{ formula: } 5n(e + e') + ee' + 5^2n^2.$$

By way of illustration, take the problem: How many are 12×13? Hold up each hand clenched. Extend the right thumb and index finger (= 12). Extend the left thumb, index finger and middle finger (= 13). How many fingers are extended? Five. $5 \times 10 = 50$. How many are extended on each hand? Two on the right hand and three on the left. $2 \times 3 = 6$. $50 + 6 + 100 = 156$.

The modulus 5 is natural under the influence of the decimal system. However, it is readily possible to substitute 4, 6, 10, or even other numbers. The modulus 6 is illustrated in

TABLE III.

1st cycle (7···12) formula: $12(e + e') + cc'$.
2d cycle (13···18) formula: $18(e + e') + cc' + 108$.
3d cycle (19···24) formula: $24(e + e') + cc' + 288$.
4th cycle (25···30) formula: $30(e + e') + cc' + 540$.
5th cycle (31···36) formula: $36(e + e') + cc' + 864$.
. . . . . .
nth cycle $[6n + 1 \cdots 6(n + 1)]$ formula: $6(n + 1)(e + e') + cc' + 6^3(n^3 - 1)$.

First cycle problem: How many are 8×9? Hold up each hand clenched. Extend the right thumb and index finger (= 8). Extend the left thumb, index finger and middle finger (= 9). [In order to use this table, one must imagine that he has two little fingers on each hand. The fingers then beginning with the thumb have values respectively of 7, 8, 9, 10, 11, 12.] How many fingers are extended? Five. $5 \times 12 = 60$. How many are closed? Four on the right hand and three on the left. $4 \times 3 = 12$. Since $60 + 12 = 72, 8 \times 9 = 72$.

The foregoing material may be cast in a form analogous to Table II as follows:

TABLE IV.

1st cycle $(7 \cdots 12)$ formula: $12(e + e') + cc'$.
2d cycle $(13 \cdots 18)$ formula: $12(e + e') + ee' + 144$.
3d cycle $(19 \cdots 24)$ formula: $24(e + e') + cc' + 288$.
4th cycle $(25 \cdots 30)$ formula: $24(e + e') + ee' + 576$.
5th cycle $(31 \cdots 36)$ formula: $36(e + e') + cc' + 864$.

The general formula as given in Table III is here valid, when n is an odd number; when, however, n is an even number, the formula is:

$$n\text{th cycle } [6n + 1 \cdots 6(n + 1)] \text{ formula: } 6n(e + e') + ee' + 6^2n^2.$$

Latin passages similar to those discussed above are as follows:

Plautus: *Stich.*, 706. Ovid: *Pont.*, 2, 3, 117. Seneca: *Epist.*, 88, 10. Pliny: *Natural History*, 2, 87. Irenæus: 5, 30, 1. Tertullian: *Apologeticus*, 19. Martianus Capella: 2, 102; 7, 729 and 746. Firmicus: *Mathesis*, 1, 4, 13. Ambrose: *Tob.*, 7, 25. Pacianus: *Epist. ad Sym. tertia*, 3, 25. Macrobius: *Saturnalia*, 1, 1, 6; *Somnium*, 2, 11, 17. Augustine: *Sermo.* (ed. A. Mai; nova patrum bibliotheca, I), 158, 14; 270, 7. Cassianus: *Conl.*, 24, 26, 7. Boethius: *In Porphy. Comm.*, Sec. 1, 2, p. 138, 19 (Migne, 64).

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A SIMPLE PROOF OF HART'S THEOREM.

By J. L. COOLIDGE, Harvard University.

The theorem which forms the subject of the present note was discovered by Sir Andrew Hart[1] and may be stated as follows:

If a triangle be formed by the arcs of three circles, the inscribed and the three escribed circles are all tangent to a new circle or line.

This theorem is to be found in numerous works on modern elementary geometry and has been proved also in countless shorter articles. All of the proofs with which I am familiar, at least all of those which are elementary in character, are essentially alike and depend upon Casey's criterion for four circles tangent to a fifth. If the circles be c_1, c_2, c_3, c_4, a necessary and sufficient condition that they be tangent to a circle or line is that there should exist among their common tangents a relation

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0,$$

where t_{ij} means the length of a common tangent to c_i and c_j.[2] Hart's theorem comes very quickly by four successive applications of this relation. The trouble is that the amount of confidence that one is willing to repose in this criterion varies inversely as the amount of care with which one regards it. The difficulties are as follows:

(*a*) The proof that this equation affords a sufficient condition that four circles should be tangent to a fifth is rather intricate.[3]

(*b*) The condition, as a theorem in elementary geometry, is not necessary. We may have four circles tangent to a fifth where one of the four surrounds the other three. Here, three of the common tangents simply do not exist in the universe of discourse.

(*c*) Some of the common tangents may be direct and some transverse, and exact statement covering all cases correctly is cumbersome.

(*d*) It is a tedious task to look after all of the + and − signs. The usual method is to write them all + and trust in Providence.

It is for these reasons that it seems to me worth while to give another proof based upon different considerations. Our method consists simply in throwing Hart's theorem back upon the theorem of Feuerbach, and assuming that the reader is familiar with one, at least, of the many ordinary proofs of the latter.[4] Feuerbach's theorem states that the inscribed and escribed circles of a triangle are tangent to a fourth circle, namely, the nine-point circle.

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[1] "On the Extension of Terquem's Theorem," *Quarterly Journal*, Vol. IV, 1860.

[2] Conf. Casey's *Sequel to Euclid*, second ed., London, 1881, p. 101.

[3] Conf. Lachlan, *Modern Pure Geometry*, London, 1893, p. 244. Casey avoids the difficulty by assuming that a necessary condition must be sufficient.

[4] Certain mathematicians seem to consider that the proving of Feuerbach's theorem constitutes a separate branch of mathematics. Conf. Sawayama, "Nouvelles démonstrations d'un théorème relatif au cercle de neuf points," *L'Enseignement mathématique*, Vol. XIII, 1911. This paper contains nine new proofs.

Proof. The circles constituting the triangle shall be $c_1c_2c_3$, the inscribed circle c, while the escribed circles need not be named. Let c' be the circle tangent to c_1, c_2, c_3 which couples with c, *i. e.*, has either exactly the same contacts with all three, or exactly the opposite contacts. If c' reduce to a point, we may at once invert c_1, c_2, c_3 into three lines, and we have reached Feuerbach's theorem. If c' be not a point or a straight line (a case which we may avoid by a preliminary inversion) let N be a point on the axis of this circle at a radius's distance from the center. We take a sphere of inversion with N as center, passing through c'. The plane will invert into that sphere which has N for north pole and c' for equator, and our problem consists in proving Hart's theorem for the sphere, where the given circles touch the equator. We next make the simple transformation which consists in replacing each great circle by one of its poles, say that which lies in the northern hemisphere. This transformation carries a circle into a circle, and tangent circles into tangent circles. Hence we have merely to prove Hart's theorem on a sphere where the three given circles pass through the north pole. But if we repeat our previous inversion, the three circles through the north pole become three straight lines in the plane, and we are carried back to Feuerbach's theorem once more. Our proof of the dependence of the one theorem upon the other is, thus, complete.

A TRIBUTE TO JOHN HOWARD VAN AMRINGE.

In the death of Professor John Howard Van Amringe September 10, 1915, in his eighty-first year, there passed away from Columbia College one of the greatest men, and without doubt the most beloved man, ever connected with that venerable institution.

Born of Dutch parentage in Philadelphia in 1836 and prepared for college mainly by his father, he entered Yale in 1854. At the end of his sophomore year he transferred to Columbia College and was graduated from this institution with the degree of A.B. in 1860, at the age of twenty-four. So brilliant and many-sided were his native powers and his attainments that even before graduation he had been tendered an instructorship in no fewer than five widely diverse departments: Greek, Latin, history, chemistry, mathematics. He might with equal propriety have been invited into the department of English, had there been such a department at that time, for his extant writings, including many published addresses, show that he had a remarkable control over the resources of English speech.

He chose mathematics and this subject he taught till his resignation from Columbia University, October, 1909, after fifty years of service as teacher and many years of service as dean of the college. Why he chose this science I do not know, but, as President Butler has felicitously said, "There was something curiously appro priate in his choice of mathematics as the agency of his academic influence, for there was in it that rigor of demonstration and that accuracy of statement which marched so well with his sturdy uprightness, his straight

thinking, and his unbending integrity; and there was also in it that beauty of expression, that perfection of form, that appeal to the imagination which reflected his literary power, his literary appreciation, and his splendid hold on written speech."

Van Amringe was a great teacher, especially of undergraduate men. He was never converted to a belief in coeducation. He did not believe in, and he did not employ, the lecture method with undergraduates. He was convinced that one of the great desiderata is to teach students to read solid books understandingly, and so he assigned them daily definite lessons in a chosen book and required them to report, usually in the form of classroom recitations. Idleness was not tolerated; industry and achievement were praised generously and discriminatingly. If a student, having tried, failed to understand, he was not overwhelmed by explanations, but was led, by the too rare art of skillful questioning and suggestion, into the presence of the truth. But it was not his method nor his subject that made Van Amringe a great teacher. The secret lay in his personality, deep, rich, sympathetic, generous, upright, sturdy and profoundly devoted to the higher things of the spirit. What Professor Van Amringe taught was men. With him mathematics was not an end; it was a means, an agency for making his presence and personality effective in the enlightenment and edification of young men. And he had his reward: the great and increasing gratitude, admiration, and love of generation after generation of students and colleagues, manifested continuously and everywhere in private and in public. Despite all this he preserved throughout that modesty, simplicity and democracy of spirit that we like to fancy essential in the character of a great man. Indeed, his successor in the office of Dean, Dr. Keppel, has justly characterized "Van Am" as "absolutely unspoilable."

Of technical mathematical ability, Professor Van Amringe had much. Fortunately, he did not devote it to mathematical research and publication, for, had he done so, he could not have rendered the manifold other services which he did render and which but few producers of mathematics are qualified to render. He did indeed edit a now obsolete series of the mathematical works of Davies, and in 1874 he published a valuable pamphlet on "The Theory and Practice of Life Insurance." But, in the main, his great abilities expressed themselves in other forms of service: as a powerful helper in the establishment and direction of the Columbia School of Mines; as a contributor towards the establishment and early conduct of graduate mathematical instruction in Columbia University; as first president, and subsequently as member of the council, of the American Mathematical Society; as the author of a history of Columbia College; as churchman and publicist in the matter of good counsel, especially in the management of hospitals and schools; as chief organizer, leader and stimulator of alumni associations; but above all and most joyously of all as an exemplar for a half century of how mathematics may be made the means of releasing the faculties of young men and winning their allegiance to rectitude of thought and life. Professor and Dean Van Amringe was one of the greatest figures in the collegiate history of American education.

C. J. KEYSER.
COLUMBIA UNIVERSITY.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Plane Analytic Geometry. By MAXIME BÔCHER. Henry Holt and Company, New York, 1915. xiii + 235 pages. $1.60.

In reading this book the first thing to impress one is the pleasing style. Everything is so simply and clearly, yet at the same time so accurately stated, that the average student should be able to read the text by himself and understand even the finer points. It is exactly the sort of book that those who have been fortunate enough to hear Professor Bôcher's lectures might have expected him to write. To give an example of this accuracy of statement, in deriving the equation of a locus, the author points out very clearly that all we have shown is that the moving point must satisfy the derived equation, and hence that the required locus may form only a part of what we have found. Again, he is at some pains to show that for the ellipse, when we rationalize the equation $FP + F'P = 2a$, the resulting equation is really equivalent to the original one—something the reviewer remembers seeing in no other text. Then, too, in deriving the equations of a straight line, care is taken to show, not merely that any point on the line satisfies the equation, but also the converse. Something, again, that is not done in every text.

In the introduction it is stated that "the one aim should be to put the student into possession of an instrument which he can himself use in proving new geometrical theorems or solving new problems." As a result of the observance of this principle, many will feel that there are too many theorems to be proved analytically and not enough drill problems. Whether this feeling is justified or not depends, of course, on the class and the instructor.

Here, as well as in the author's *Trigonometry*, a great deal of matter has been put in fine print to be omitted by the average class. The principal topic so relegated, besides oblique coördinates, is the normal form of the equation of a straight line. The distance from a point to a line is found by a direct method, and the question of the sign of this distance is treated later in fine print. Many will doubtless disagree with this procedure, but it has much to recommend it. The usual treatment of this distance seems a very roundabout and artificial one to students and usually presents difficulty; in fact the normal form often seems to them a most abnormal one. Omitting it tends to make the work seem more direct. On the other hand, the normal form is the only one to which the equation of *any line whatever* may be reduced, and it is almost indispensable in any work using abridged notation. This does not form an argument, however, for its being studied in a first course; and the reviewer is in hearty agreement with the arrangement in the text.

At the end of the book are two chapters on the differential calculus—not "hashed fine," as the author says in the introduction, "but put squ ly as a new subject," covering a surprisingly large amount of the subject with unusua clearness. While these chapters are very carefully written and form an excellent

introduction to the calculus, there is plenty of material in the rest of the book for those who prefer not to take up the calculus at this time.

Many who teach analytic geometry to students of whom it is required, will think the book too difficult, but others will find it well adapted to their needs. In any case, it is a book to be considered in choosing a text, and presents a distinct advance over the usual textbook.

The general appearance and typography of the book are excellent.

It should be added that Professor Bôcher has written a syllabus for a course in solid analytic geometry following the same lines as this book.

ELIJAH SWIFT.

UNIVERSITY OF VERMONT.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

[Send all Communications to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

ALGEBRA.

448. Proposed by W. D. CAIRNS, Oberlin College.

In the *Washington* (D. C.) *Times*, Mr. W. A. Dayton called attention some weeks ago to a curious repetition of digits in the decimal value of 1/115. If this decimal, which we print in the form $0.\dot{8}6956521739130\ 43478260\dot{}$ be divided by two, the result is $0.\dot{4}3478260\ 86956521739130\dot{}$, the fourteen-digit and eight-digit groups having been thus interchanged. A similar result, as he points out, is obtained if the original decimal value is divided by four. Mr. Dayton asks that this curiosity be explained.

449. Proposed by FRANK IRWIN, University of California.

Sum the expression

$$1+2\binom{k+1}{k}+3\binom{k+2}{k}+\cdots+(n-k+1)\binom{n}{k}.$$

Also show how to sum

$$1\cdot2+2\cdot3\binom{k+1}{k}+3\cdot4\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)\binom{n}{k},$$

$$1\cdot2\cdot3+2\cdot3\cdot4\binom{k+1}{k}+3\cdot4\cdot5\binom{k+2}{k}+\cdots+(n-k+1)(n-k+2)(n-k+3)\binom{n}{k},$$

etc., where $\binom{l}{k}$ is used to denote the coefficient of x^k in $(1+x)^l$.

GEOMETRY.

479. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the locus of the point whose polars (polar planes) with respect to two given conics (quadrics), are perpendicular to each other.

480. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Of equal quadrilaterals on the same base, that which has the least perimeter must have the angles not adjacent to the base equal to each other.

CALCULUS.

400. Proposed by H. S. UHLER, Yale University.

The axis of a prism whose right-section is a regular polygon of apothem a and of n sides passes through the center of a sphere of radius R. Show that, in general, the volume may be expressed by the formula:

$$V = \frac{4}{3}\pi R^3 + \frac{2}{3}a^2 n\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n} + \frac{1}{3}an(3R^2 - a^2)\sin^{-1}\left[\frac{2a\left(R^2 - a^2\sec^2\frac{\pi}{n}\right)^{1/2}\tan\frac{\pi}{n}}{R^2 - a^2}\right]$$

$$-\frac{4}{3}nR^3\sin^{-1}\left[\frac{R\sin\frac{\pi}{n}}{(R^2 - a^2)^{1/2}}\right].$$

Also discuss the two special cases where

$$a = R\cos\frac{\pi}{n}, \text{ and } n = \infty.$$

401. Proposed by LAENAS G. WELD, Pullman, Illinois.

Given a continuum of triangles whose sides are in arithmetical progression, the common difference being h: (a) The ratio of the mean value of the areas of all the triangles, the mean of whose three sides is not greater than μ, to the area of the triangle, the mean of whose three sides is equal to μ, is $(\mu + 2h)/3\mu$. Indicate the limiting values of this ratio and show that, when it is equal to $\frac{1}{2}$, the triangle whose mean side is μ is right angled. (b) The ratio of the mean value of the areas of the circles inscribed in all of these triangles, the mean of whose three sides is not greater than μ, to that of the circle inscribed in the triangle, the mean of whose three sides is equal to μ, has the limiting values $\frac{1}{2}$ and $\frac{1}{3}$. When the triangle whose mean side is μ is right angled, the ratio in question is 4/9. (c) Of the circles circumscribed about these triangles the minimum has the radius $2h$.

MECHANICS.

319. Proposed by LAENAS G. WELD, Pullman, Illinois.

A hexagonal pencil lies upon the inclined top of a drawing table and is on the point of either rolling or sliding. Find the angle between its direction and the horizontal edge of the table, the coefficient of friction being μ.

320. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

A heavy uniform chain of length l is hung over a rough horizontal cylinder of radius r. Show that one end of the chain will be

$$\frac{2\mu r}{\mu^2 + 1}(e^{\mu\pi} + 1) + l(e^{\mu\pi} - 1)$$

units lower than the other, just when the chain begins to move, the coefficient of friction being μ.

NUMBER THEORY.

237. Proposed by NORMAN ANNING, Chilliwack, B. C.

Prove that for three numbers, x, y, z,

$$9\Sigma(x - y)^4 = \Sigma(2x - y - z) = 2\square.$$

238. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Determine the rational value of x that will render $x^3 + px^2 + qx + r$ a perfect cube. Apply the result to $x^3 - 8x^2 + 12x - 6$.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

436. Proposed by WALTER H. DRANE, Cumberland University, Tenn.

The product of two numbers, p and q, may be obtained by dividing p by 2 successively, discarding remainders, until the quotient 1 is obtained and then multiplying q by 2 successively and adding those products, 2^iq, which correspond to those quotients of p that are odd numbers.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

The meaning of the proposition is a little clearer if we write "those products, 2^iq, $(i = 0, 1, 2, \ldots)$ which correspond to the odd numbers of the set composed of p and the quotients."

We have:

$$p = x_0 + 2n_0, \text{ where } x_0 = 1 \text{ or } 0 \text{ according as } p \text{ is odd or even;}$$

$$n_0 = x_1 + 2n_1, \text{ where } x_1 = 1 \text{ or } 0 \text{ according as } n_0 \text{ is odd or even;}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$n_{i-1} = x_i + 2n_i, \text{ where } x_i = 1 \text{ or } 0 \text{ according as } n_{i-1} \text{ is odd or even,}$$

and so on until we reach $n_k = 1$, so that $n_{k-1} = x_k + 2$.

Substituting,

$$p = x_0 + 2x_1 + 2^2x_2 + 2^3x_3 + \cdots + 2^ix_i + \cdots + 2^kx_k + 2^{k+1}$$

$$qp = x_0(q) + x_1(2q) + x_2(2^2q) + \cdots + x_i(2^iq) + \cdots + x_k(2^kq) + 2^{k+1}q.$$

This, in view of the definition of the x's, proves the proposition, for the process associates

$$(p, q), (n_0, 2q), (n_1, 2^2q), \cdots, (n_{i-1}, 2^iq), \cdots.$$

Also solved by N. P. PANDYA, S. A. JOFFE, A. H. HOLMES, FRANK IRVING, and NORMAN ANNING.

437. Proposed by C. N. SCHMALL, New York City.

Given that $s_1, s_2, s_3, \cdots, s_k$ are the sums of k arithmetical series, each taken to n terms. The first terms are, respectively, 1, 2, 3, $\cdots$, k, and the common differences are 1, 3, 5, $\cdots$ $(2k - 1)$. Show that

$$s_1 + s_2 + s_3 + \cdots + s_k = \frac{nk(nk + 1)}{2}.$$

SOLUTION BY N. P. PANDYA, Sojitra, India.

We have

$$s_k = k + \{k + (2n - 1)\} + \{k + 2(2n - 1)\} + \cdots + \{k + (n - 1)(2n - 1)\}$$

$$= \frac{2n^2k - n^2 + n}{2}.$$

Hence,

$$\sum_{i=1}^{i=k} s_i = n^2 \sum_{i=1}^{i=k} i - \frac{kn^2}{2} + \frac{kn}{2} = \frac{n^2k(k+1) - kn^2 + kn}{2} = \frac{nk(nk+1)}{2}.$$

438. Proposed by WALTER C. EELLS, U. S. Naval Academy, Annapolis, Maryland.

In Hardy's *Pure Mathematics* (page 14, Nos. 2, 3) occurs the problem: "Show that if m/n is a good approximation to $\sqrt{2}$, then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions, *e. g.*, 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, $\cdots$." Find (*a*) other approximations for $\sqrt{2}$ of the same type, *i. e.*,

$$\frac{m'}{n'} = \frac{am + bn}{cm + dn}, \ (a, b, c, d, m, n, \text{integers}).$$

(*b*) Similar approximations for the square roots of other integers.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Assume that $(m/n) < \sqrt{2}$. We wish to find a, b, c, d so that

$$0 < \frac{am + bn}{cm + dn} - \sqrt{2} < \sqrt{2} - \frac{m}{n}. \tag{1}$$

Assume a, b, c, d positive. The second inequality leads, after simple reductions and squaring both sides, to

$$c^2m^4 + 2(ac + cd)m^3n + (a^2 + 2ad + d^2 + 2bc - 8c^2)m^2n^2 + 2(ab + bd - 8dc)mn^3 + (b^2 - 8d^2)n^4 < 0. \tag{2}$$

If we had assumed $m/n > \sqrt{2}$, inequality (1) would have been reversed, and hence (2) also. But if the right-hand side of (2) changes sign with $m - \sqrt{2}\,n$, it must have $m - \sqrt{2}\,n$ and hence also $m^2 - 2n^2$ as a factor. This leads to the following equations connecting a, b, c, d.

$$ab + bd + 2ac - 6cd = 0. \tag{3}$$

$$b^2 - 6d^2 + 2a^2 + 4ad + 4bc - 12c^2 = 0. \tag{4}$$

These equations have for a solution $a = d$, $b = 2c$; and this is the only solution in positive integers.

The first inequality of (1) leads, after substitution of a for d, and $2c$ for b, to $(a^2 - 2c^2)(m^2 - 2n^2) > 0$. Hence, $a^2 < 2c^2$. Finally, in (2) putting $a = d$, $b = 2c$, we obtain, after factoring, $[m^2 - 2n^2][c^2m^2 + 4acmn + (4a^2 - 2c^2)n^2] < 0$. Hence, $c^2m^2 + 4acmn + (4a^2 - 2c^2)n^2 > 0$, or $(cm + 2an)^2 > 2c^2n^2$, that is, $m/n > \sqrt{2} - (2a/c)$. If $(2a/c) \geqq 1$, m/n need only be as large as .5 to satisfy this inequality.

If we had assumed $m/n > \sqrt{2}$, we should obviously be led to the same results.

If, then, $m'/n' = (am + 2cn)/(cm + an)$, and $2a \geqq c$, $a^2 < 2c^2$, or in other words $8c^2 > 4a^2 \geqq c^2$, we shall have satisfied the conditions of the problem. Such

sets of values are $\begin{cases} c=1 \\ a=1 \end{cases}, \begin{cases} c=2 \\ a=1 \end{cases}, \begin{cases} c=3 \\ a=2 \text{ or } 4 \end{cases}$, etc. The first set is the one given in the problem. The second gives $m'/n' = (m+4n)/(2m+n)$ which yields the series 1/1, 5/3, 17/13, 69/47, ...

(*b*) Almost the same work leads to similar results in the case of any surd, $\sqrt{k}$. We find $a = d$, $b = kc$, $2a \geqq c(k-1)$, $kc^2 > a^2$. These possess, among others, the following solutions for the case where $k = 3$.

$$\begin{cases} c=1 \\ a=1 \end{cases}, \begin{cases} c=2 \\ a=3 \end{cases}, \begin{cases} c=3 \\ a=4 \text{ or } 5 \end{cases}, \text{ etc.}$$

The first set gives $m'/n' = (m+3n)/(m+n)$, which yields the series 1/1, 4/2 = 2/1, 5/3, 14/8 = 7/4, 19/11, ...

Also solved by NORMAN ANNING.

GEOMETRY.

466. Proposed by HORACE OLSON, Chicago, Illinois.

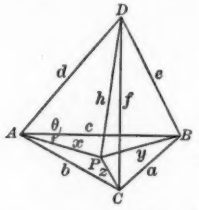
Given the edges of a triangular pyramid, find the radius of the inscribed sphere.

SOLUTION BY A. H. HOLMES, Brunswick, Maine.

Calling the radius of the sphere r, and using the notation in the figure, we have,

$$h^2 = d^2 - x^2 = c^2 - y^2 = f^2 - z^2,$$

$$\cos\theta = \frac{c^2 + x^2 - y^2}{2cx} = \frac{c^2 + d^2 - e^2}{2cx} = \frac{n}{x}. \qquad (x\cos\theta = n.)$$



$$\cos(A-\theta) = \cos A\cos\theta + \sin A\sin\theta = \frac{b^2 + x^2 - z^2}{2bx} = \frac{b^2 + d^2 - f^2}{2bx} = \frac{m}{x}.$$

Hence,

$$\sin\theta = \frac{m - n\cos A}{x\sin A}, \qquad x\sin\theta = \frac{m - n\cos A}{\sin A}.$$

Hence,

$$x^2 = n^2 + \frac{(m - n\cos A)^2}{\sin^2 A}, \qquad \text{and} \qquad h = \frac{\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2}}{\sin A}.$$

Hence,

$$3 \text{ times contents of pyramid} = \frac{bc}{2}\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2},$$

and therefore,

$$r = \frac{bc\sqrt{(d^2 - n^2)\sin^2 A - (m - n\cos A)^2}}{2(ABC + ACD + ABD + BCD)},$$

in which

$$m = \frac{b^2 + d^2 - f^2}{2b}, \qquad n = \frac{c^2 + d^2 - e^2}{2c}, \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\sin A = \sqrt{1 - \frac{(b^2 + c^2 - a^2)^2}{4b^2c^2}}, \qquad ABC = \sqrt{s(s-a)(s-b)(s-c)},$$

and similarly for *ABD*, *ACD* and *BCD*.

Also solved by WALTER C. EELLS and J. W. CLAWSON.

CALCULUS.

380. Proposed by C. N. SCHMALL, New York City.

Show that

$$\int_0^\infty \left[\frac{1}{1^4 + x^2} + \frac{1}{2^4 + x^2} + \frac{1}{3^4 + x^2} + \cdots\right] dx = \frac{\pi^3}{12},$$

where the series in the brackets is infinite.

SOLUTION BY A. M. HARDING, University of Arkansas.

For all values of x in the interval $(0, \infty)$ the nth term of the given series has the property

$$\frac{1}{n^4 + x^2} \leqq \frac{1}{n^4}.$$

Now the series $\sum_{n=1}^{\infty} 1/n^4$ converges. Hence, the given series is uniformly convergent in the interval $(0, \infty)$.

Each term of the series is continuous in the interval $(0, \infty)$.

Hence, it may be integrated term by term. Hence,

$$\int_0^\infty \left[\frac{1}{1^4 + x^2} + \frac{1}{2^4 + x^2} + \frac{1}{3^4 + x^2} + \cdots\right] dx = \frac{\pi}{2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right].$$

It can be easily shown that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Hence, the given integral $= \dfrac{\pi^3}{12}$.

Also solved by S. A. JOFFE, and J. A. CAPARO.

299. Proposed by B. F. FINKEL, Drury College.

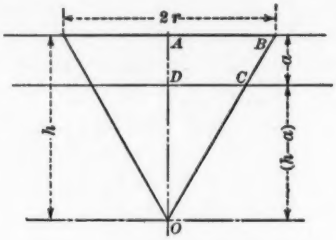
A cone rests in two fluids which do not mix, with its vertex downwards and its base in the surface of the upper fluid; to find how much its density must be increased that it may rest with its base in the common surface of the fluids. (From Walton's *Hydrostatical Problems.*)

SOLUTION BY J. F. BRACHO, University of Notre Dame.

Let, w = density of cone in first position, w_1 = density of cone in second position, d_1 = density of upper fluid, d_2 = density of lower fluid. We have:

$$DC = \frac{AB + OD}{OA} = \frac{r(h-a)}{h}.$$

By Archimedes' principle, we may write,



$$\frac{1}{3}\pi r^2 hw = \left[\frac{1}{3}\pi r^2 h - \frac{1}{3}\pi \frac{r^2(h-a)^3}{h^2}\right] d_1 + \frac{1}{3}\pi \frac{r^2(h-a)^3}{h^2} d_2;$$

Then,

$$w = d_1 + \frac{(h-a)^3}{h^3}(d_2 - d_1).$$

For the second position of the cone we must have,

$$w_1 = d_2.$$

Hence, increase of

$$w = w_1 - w = d_2 - d_1 - \frac{(h-a)^3}{h^3}(d_2 - d_1) = (d_2 - d_1)\left[1 - \frac{(h-a)^3}{h^3}\right].$$

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

At the time of making up copy for this issue no replies had yet been received for Questions numbered 4, 8, 12, 13, 16, 20 and 28.

REPLIES.

27. A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytic geometry and differential and integral calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours?

Note.—Last month we published a reply to this question by a professor in a middle western college. The reply given below was received while the December issue was still in press. It therefore gives an entirely independent view by an instructor in an eastern university. We hope that our readers will compare the two replies and note the points of agreement.—EDITOR.

REPLY BY R. B. ROBBINS, Sheffield Scientific School, Yale University.

To deal intelligently with the question it must be made more definite. Let us assume that just one year of calculus is presupposed and that an additional twelve-hour curriculum is desired which can be repeated without much modification year after year. With only the interests of prospective students of pure mathematics in mind it seems that such a college might hope to attain two fairly definite ends (I and II below) by means of the courses outlined below:

I. Introduce the student to the general subject matter of mathematics by courses in (*A*) topics in analysis—3 hours; (*B*) modern geometry—3 hours.

II. Give the student some of the standard tools for more advanced work by courses in (*C*) advanced calculus and solution of differential equations—3 hours; (*D*) infinite series, definite integrals and topics in advanced algebra—3 hours.

This doubtless looks rather formidable for a third-year college student; therefore a few words of explanation. In his junior year the student should take courses *B* and *C*, reserving courses *A* and *D* for his senior year.

Course *C* might well contain quite a variety of material. If the student does not have access to a course in mechanics in some other departments it certainly should contain some of the applications of the calculus to mechanics. This is the place to make partial differentiation and certain topics in the integral calculus concrete by application to solid analytic geometry. The solution of the commoner types of differential equations should also be included here.

Course *B* can be made as difficult and dull or as easy and inspiring as the instructor desires. If the student is led gradually into the meaning and importance of the transformations involved and is led to see why homogeneous coördinates are so useful, he will appreciate the power of his methods and will see in them a genuine beauty probably new to him in mathematical experience.

Courses *A* and *D* can best be studied together. In course *A* should be included an introduction to the theory of functions of a real and of a complex variable. It might also contain some of the easier parts of the theory of ordinary differential equations. Course *D* will include material useful in course *A* and in fact in any further work in pure mathematics. The importance of this course as an introduction to further work in mathematics cannot be over-emphasized and yet the material can be grasped by a college senior whose ability justifies continuing the study of mathematics.

An immediate objection to such a set of courses is the difficulty of finding text-books. Course *C* would not be so troublesome in this respect. As for course *B*, it seems to the writer that it is time for a book to appear dealing with methods in modern geometry from the bottom up (from the student's point of view) rather than from the top down. The material of courses *A* and *D* is such that if the instructor knows it well enough to teach it without a book, the student will be much better off if he makes his own text-book from careful notes; while if the instructor does not know it thoroughly, a text would be of little value and he better not attempt to give the courses.

There is room for much more coöperation than at present exists between the colleges of the type in question and the graduate schools in the study of mathematics. Since the only equipment essential for the mathematics department is the instructor and the blackboard, there is no reason why first class introductory work in advanced mathematics should not be done in a great many of our colleges. On the other hand, the demand for graduate instruction in mathematics is so small and the opportunities at the few large graduate schools are such that it is rarely wise for a graduate student to study at a place where he commands the

attention of very few men who are giving their best efforts to research in mathematics. For this reason, the type of college here considered has a most important work in opening the eyes of the student to the possibilities ahead of him and in giving him definite, essential tools with which to work; but such a college blunders if, instead, it tries to give more specialized courses which demand for their full appreciation a preparation which the student cannot have had.

29. While studying the problem of two equal rough bodies, connected by an inelastic wire, resting on an inclined plane, Professor Clifford N. Mills of South Dakota State College met the following interesting expression. If $1/a$ and $1/(a+1)$ are the coefficients of friction, the tension of the wire when the bodies are about to descend becomes a multiple of $\frac{1}{2}/[1/a + 1/(a+1)]$. This, when simplified, becomes $(2a+1)/2a(a+1)$. If $2a+1$ and $2a(a+1)$ represent the base and altitude of a right triangle, the hypothenuse is $2a^2+2a+1$. Therefore, this gives a series of numbers which satisfy the relation $x^2+y^2=z^2$, if a is given any value whatsoever. Professor Mills desires to know if this will give all the integers which satisfy the condition that the sum of the squares of two integers equals the square of an integer.

Reply by S. Lefschetz, University of Kansas.

If the integers x, y, z, form a Pythagorean set, that is, if they satisfy the relation $x^2+y^2=z^2$, then it is well known* that there can be found three other integers m, n, p, such that $x=p(m^2-n^2)$, $y=2pmn$, $z=p(m^2+n^2)$. For $m=a+1$, $n=a$, $p=1$ the special solution indicated by Professor Mills is obtained.

NOTES AND NEWS.

Edited by D. A. Rothrock, Indiana University.

Professor Paul Painlevé, of the department of mathematics in the University of Paris, is minister of education in the present French cabinet.

Henry Holt and Company have just published a "Plane Analytic Geometry," by Professor Maxime Bôcher. The last two chapters are devoted to calculus.

Ginn and Company have published "Problems in the Calculus," by Dr. D. D. Leib of the Sheffield Scientific School, Yale University, also "The Theory of Invariants," by Professor O. E. Glenn, of the University of Pennsylvania.

The sixteenth in the series of Wiley's Mathematical Monograph Series appears as "Diophantine Analysis," by Professor Robert D. Carmichael. The same firm has recently published an "Analytic Geometry," by Professor H. B. Phillips, of the Massachusetts Institute of Technology.

Junior Professors Peter Field, L. C. Karpinski and T. R. Running have been promoted to associate professorships of mathematics at the University of Michigan, and Drs. T. Fort and T. H. Hildebrandt to assistant professorships

* Cf. Bachman, *Zahlentheorie*, Vol. I, p. 192. Also Carmichael, *Diophantine Analysis*, pp. 8–13.

of mathematics. DR. A. L. NELSON has been appointed to an instructorship in mathematics.

In *Science* for October 22 appears a paper by PROFESSOR C. N. MOORE in which he points out the danger of assuming that the coefficient of correlation is necessarily a satisfactory measure of all forms of relationship between two variable quantities, at the same time suggesting a method of attack for determining in what way a particular relationship depends on the value of this coefficient.

The United States Bureau of Education has recently published a bulletin entitled "Mathematics in the lower and middle commercial and industrial schools of various countries represented in the International Commission on the Teaching of Mathematics." It is furnished to teachers of mathematics on application to the Bureau.

"Cubic surfaces and their nodes" is the title of an article by Dr. S. LEFSCHETZ in the *Bulletin* of the University of Kansas, Volume IX, number 6. This bulletin is issued at irregular intervals. The present volume contains 290 pages and 76 plates. It is a science number, containing 21 articles, the one mentioned above being the only one on a mathematical subject. The exchange editor is Dr. U. G. MITCHELL, of the MONTHLY staff.

PROFESSOR H. L. RIETZ, University of Illinois, was appointed a member of a committee of four to investigate the operation of all pension laws enacted in the State of Illinois, "together with the present and future cost thereof, and to collect information as far as possible in regard to the operation of similar laws in other states, and countries, and to make recommendation upon this subject to the next General Assembly."

At the High School Conference held at the University of Illinois on November 18–20, the following mathematical papers were presented: "Algebra from the utilitarian standpoint," by Dr. A. R. Crathorne; "The experimental determination of standards in first year algebra," by Dr. H. O. Rugg; "Report of the committee investigating high school mathematical libraries," by Professor E. H. Taylor; "Graphs in elementary algebra," by Mr. H. C. Zeis; "The function notion in elementary algebra," by Professor J. F. Millis.

The part of the *Encyclopédie des Sciences Mathématiques* which treats the theory of domains of algebraic numbers appeared on June 18, 1915. The first three pages conclude the article on transcendental propositions relating to the theory of numbers and the last seven pages begin the article on complex multiplication. The remaining 86 pages are devoted to algebraic number realms, while the German edition devotes only about 23 pages to this subject. The present part is issued as Tome I, volume 3, fascicule 5, and consists of 96 pages.

The third regular meeting of the Association of Mathematics Teachers of New Jersey was held at Stevens Institute of Technology, November 20, under the Presidency of DEAN HENRY B. FINE. Papers were read as follows: "Mathematics and insurance," by P. C. H. PAPPS; "The proper functioning of a high school course in geometry," by R. T. LE VALLEY: "A review of Bourlet's plane geometry," by B. B. STRANG; and "A high school course in strength of materials," by G. D. ORNER.

The Mathematical Club of the University of Illinois is divided into two sections, known as the Graduate Section and the Undergraduate Section, respectively. The former meets bi-weekly for the consideration of papers involving new results or new methods, while the latter meets monthly for the consideration of questions of general mathematical interest and the solutions of problems. Each meeting lasts about one hour. The Undergraduate Section usually arranges for one social evening meeting annually. For the current year Professor G. A. Miller has been elected chairman of the Graduate Section, and Mr. G. W. Smith, chairman of the Undergraduate Section.

A "mathematics contest" which was held between the Hyde Park High School and University High School of Chicago is described by Mr. RALEIGH SCHORLING in *School Science and Mathematics* for December. In the same issue appear "The proofs of the law of tangents," by R. M. Mathews, "A simple and effective method of solving a polynomial," by C. H. FORSYTH, a "Graphical method for cubic equations," by ALFRED RITTER, and "Historical notes in textbooks on secondary mathematics," by PROFESSOR G. A. MILLER, the last-named article criticizing the accuracy of the historical references made in a recently published high school text.

The Mathematics Club of the University of Kansas is an organization of students with Dr. E. B. STOUFFER as faculty adviser. It was started in 1911 and now has twenty-six members. The Club meets bi-weekly and is carrying out the following program during the present year: Fermat's theorem and allied topics, Dr. S. LEFSCHETZ; "Non-Euclidean geometry," Miss JESSIE JACOBS; "Line construction," Miss ADA WEST; "Who's who in mathematics in America," Dr. U. G. MITCHELL; "Mathematical reference books," Dr. E. B. STOUFFER; "Curve tracing," Mr. P. W. HORNLEY; "Methods of computing errors," Dr. H. E. JORDAN; "Quadratic forms in number theory," Miss WILMA ARNETTE; "Elements of orbits of heavenly bodies," Miss CORA SHINN; "Mathematical fallacies," Mr. J. B. RAMSEY; "Some definite integrals," Mr. A. W. LARSEN; "The origin of the calculus," Mr. L. L. STEIMLEY; "Finite geometry," Mr. CYRIL NELSON; "Review of *Memorabilia Mathematica*," Miss FLORENCE SCHEIDENBERGER.

Beginning with this issue the MONTHLY becomes the Official Journal of THE MATHEMATICAL ASSOCIATION OF AMERICA, and will be subject to the control of the Council of the ASSOCIATION. Thus the MONTHLY enters a broad field of

usefulness and is assured of wise and efficient support. The Constitution provides that the official journal shall be directly controlled by a Committee on Publication, consisting of a Managing Editor and two other members, who are appointed by the Council and thereby become *ex officio* members of the Council. The Editorial Board consists of the three members of the Publication Committee, together with twelve Associate Editors selected by them. The editorial work is divided among various subcommittees, and all are now laying plans for immediate aggressive action. They will welcome suggestions from all sources, looking toward the strongest possible equipment of the MONTHLY for the important responsibilities now laid upon it.

The Editorial Board for 1916 is as follows:

H. E. Slaught, *Managing Editor*, University of Chicago,
W. H. Bussey, University of Minnesota,
R. D. Carmichael, University of Illinois,
with the co-operation of
R. P. Baker, University of Iowa,
W. C. Brenke, University of Nebraska,
A. Cohen, Johns Hopkins University,
B. F. Finkel, Drury College,
L. C. Karpinski, University of Michigan,
G. H. Ling, University of Saskatchewan,
Helen A. Merrill, Wellesley College,
U. G. Mitchell, University of Kansas,
W. H. Roever, Washington University,
D. A. Rothrock, Indiana University,
C. S. Slichter, University of Wisconsin,
D. E. Smith, Columbia University.

NOTES ON THE COLUMBUS MEETING.

A most remarkable circumstance occurred at Columbus, in connection with the choosing of the name for the new organization. A committee of three had been chosen to sift the eighteen proposals and make a recommendation. The committee agreed to act independently and each to make his choice by himself. They did so and each made the same choice. Moreover, five other members, after discussing the matter informally by themselves, also came to the same conclusion. This seemed so remarkable that the final adoption seemed inevitable.

Another interesting feature of the Columbus meeting was the remarkable unanimity of purpose displayed by representatives of all interests concerned. There seemed to be no doubt as to the importance of the opportunity presented to this new organization in its chosen field. There was no lack of serious interest and of genuine enthusiasm.

Great confidence was placed in the judgment and wisdom of the nominating committee for the new Association. All will realize how difficult a task they had to perform in distributing the various positions of responsibility among the districts and interests far more numerous than the number of nominees. How well they performed this task remains to be shown by the fidelity and efficiency with which these officers discharge their duties, and by the clearness of vision which shall animate them as they contemplate their opportunities for genuine service to the cause of mathematics in America.

Particular attention is called to the second section in the By-laws on the nomination and election of officers. The success of such a thoroughly democratic plan of procedure will depend entirely upon the extent to which the members appreciate its importance and uphold its dignity by conscientious and thoughtful use of the franchise thus provided.

Many questions have been asked with reference to what will be the attitude of THE MATHEMATICAL ASSOCIATION OF AMERICA to the existing secondary associations in various parts of the country. The official answer to these questions is section 2 in the first article of the Constitution. It may be further stated unofficially that the spirit of the new ASSOCIATION, so far as it has had opportunity to manifest itself, will surely be one of friendly coöperation. It is fully recognized that these associations have most important and far-reaching problems in their own field and that they are doing efficient service toward the solution of these problems. But the new ASSOCIATION also has its peculiar problems in a field hitherto quite unoccupied, and, in entering upon its responsibilities, it invites the coöperation of all individuals, or groups of individuals, who may in any way be interested in the field of collegiate mathematics.

Blanks for admission to membership in THE MATHEMATICAL ASSOCIATION OF AMERICA are sent under separate cover to all persons to whom this issue of the MONTHLY goes. All who return these with the annual dues before April 1, 1916, will be entitled to admission without the initiation fee and will constitute the Charter Membership of the Association. How large this initial list of members shall be depends upon the readiness of response of individuals. Thus far, about seven hundred persons have manifested direct and personal interest by signing and returning cards, thus giving rise to the hope that the charter membership may reach that number. Shall it be so?

The annual dues in THE MATHEMATICAL ASSOCIATION OF AMERICA have been fixed at THREE DOLLARS, including a yearly subscription to THE AMERICAN MATHEMATICAL MONTHLY, this being the lowest possible figure at which the expenses of the ASSOCIATION can be met on a membership basis of less than one thousand. The MONTHLY alone has actually cost about this amount per subscriber for the past three years, the deficit being provided for by subsidies from fourteen universities and colleges. These subsidies will, of course, no longer be continued.

IMPORTANT NOTICE TO PRESENT MONTHLY SUBSCRIBERS.

Henceforth, the subscription price of the MONTHLY will be *three dollars net to all non-members of the* ASSOCIATION. The following adjustments for prospective members are proposed:

(1) Those who have already paid their subscriptions for the entire year 1916 are asked to send *one dollar* additional, which will entitle them to membership in the ASSOCIATION.

(2) Those who have not paid for 1916 are asked to send *three dollars*, which will entitle them to membership and include the MONTHLY. *No further subscriptions for 1916 will be received at the old rate of two dollars.*

In the case of subscriptions under (1) or (2) which expire *before* the end of 1916, please add *twenty cents extra* for each copy needed to complete the year. *Hereafter all subscriptions will date from January of each year.*

(3) An institution in which the Calculus is taught may become an *institutional member* of the ASSOCIATION by the payment of *five dollars* annually, which will entitle the library to receive *two copies of the* MONTHLY and the institution to send a voting delegate to all meetings of the ASSOCIATION. Institutions in which the Calculus is taught, whose libraries have already renewed their subscriptions for 1916, are asked to send *three dollars additional* and thus become institutional members of the ASSOCIATION.

Other institutions, and those not wishing to become institutional members, whose library subscriptions have already been renewed for 1916, are asked to send *one dollar additional* to complete the advanced price of the MONTHLY. No further subscriptions will be received at the old rate of two dollars, *and no discount from the advanced rate of three dollars will be allowed on subscriptions made through agencies.*

(4) The obligations of the MONTHLY for 1916 will, of course, be fulfilled on the former basis in the case of any individual or institution whose subscription has already been paid, and who may decline to make the adjustment on the new basis.

(5) Please note that all subscriptions to the MONTHLY and dues in the ASSOCIATION are to be paid to the SECRETARY-TREASURER, Professor W. D. CAIRNS, 55 East Lorain St., Oberlin, Ohio.

In order to make sure of receiving the February MONTHLY, please sign and return the membership blanks IMMEDIATELY. Otherwise the publishers cannot determine how large an edition to print.

IMPORTANT ANNOUNCEMENT

TO

ALL INTERESTED IN MATHEMATICAL PROGRESS

THE AMERICAN MATHEMATICAL MONTHLY, since its reorganization in January, 1913, has endeavored to fulfill its mission as "A JOURNAL FOR TEACHERS OF MATHEMATICS IN THE COLLEGIATE AND ADVANCED SECONDARY FIELDS."

A selection from the Tables of Contents thus far includes articles on—

The History of Mathematics, such as the following:

"History of the Exponential and Logarithmic Concepts," by PROFESSOR FLORIAN CAJORI of Colorado College;

"The Foundation Period in the History of Group Theory," by JOSEPHINE BURNS, Graduate Student at the University of Illinois;

"Errors in the Literature on Groups of Finite Order," by PROFESSOR G. A. MILLER, University of Illinois;

"Number Systems of the North American Indians," by PROFESSOR W C. EELLS, United States Naval Academy;

"The Algebra of Abu Kamil," by PROFESSOR L. C. KARPINSKI, University of Michigan;

"Centers of Similitude of Circles and Certain Theorems Attributed to Monge. Were they known to the Greeks?" by PROFESSOR R. C. ARCHIBALD, Brown University;

"The History of Zeno's Arguments: Phases in the Development of the Theory of Limits," by PROFESSOR FLORIAN CAJORI, Colorado College.

Pedagogical Considerations, such as the following:

The "Foreword" concerning Collegiate Mathematics, by PROFESSOR E. R. HEDRICK, University of Missouri;

"Some Things we wish to know," by PROFESSOR E. R. HEDRICK;

"Mathematical Literature for High Schools," by PROFESSOR G. A. MILLER;

"Mathematical Troubles of the Freshman," by PROFESSOR G. A. MILLER;

"Minimum Courses in Engineering Mathematics," by PROFESSOR SAUL EPSTEEN, University of Colorado;

"Incentives to Mathematical Activity," by PROFESSOR H. E. SLAUGHT, University of Chicago;

"Synthetic Projective Geometry as an Undergraduate Study," by PROFESSOR W. H. BUSSEY, University of Minnesota;

"Retrospect and Prospect," by PROFESSOR H. E. SLAUGHT;

"Note on a Memory Device for Hyperbolic Functions," by F. S. ELDER, Central High School, Kansas City, Mo.;

"A Plea for less Formal Work in Mathematics," by F. M. MORGAN, Dartmouth College;

"A Simple Algebraic Paradox," by PROFESSOR J. L. COOLIDGE, Harvard University;

"Note on Simple Algebraic Equations," by PROFESSOR H. L. SLOBIN, University of Minnesota;

"On Courses in Synthetic Projective Geometry," by PROFESSORS LAO G. SIMONS, Normal College of the City of New York, C. E. STROMQUIST, University of Wyoming, T. G. RODGERS, Normal School of New Mexico, R. D. CARMICHAEL, and D. N. LEHMER;

"On the Cultural Value of Mathematics," by PROFESSORS W. T. STRATTON, Kansas State Agricultural College, and D. N. LEHMER;

"On Courses in the History of Mathematics," by PROFESSORS W. T. STRATTON and G. A MILLER;

"Remarks on Klein's Famous Problems of Elementary Geometry," by PROFESSOR R. C. ARCHIBALD, Brown University;

"On the Trisection of an Angle and the Construction of Regular Polygons of 7 and 9 Sides," by PROFESSOR L. E. DICKSON, University of Chicago;

"An Equation Balance for Class-Room Use," by PROFESSOR E. W. PONZER, Stanford University;

"A Cardioidograph," by C. M. HEBBERT, University of Illinois;

"Coördinated Courses in High School Mathematics," by EDITH LONG, Lincoln, Neb., and ROY CUMINS, Columbia University;

"Conference Periods for Students," by PROFESSOR C. R. MCINNES, Princeton University, and PROFESSOR C. S. ATCHISON, Washington and Jefferson College;

"Determinant Formula for Coplanarity of Four Points," by PROFESSOR A. M. KENYON, Purdue University;

"What can the Colleges do toward Improving the Teaching of Mathematics in the Secondary Schools?" by PROFESSOR C. N. MOORE, University of Cincinnati.

General Mathematical Information, such as the following:

"The Third Cleveland Meeting of the American Association for the Advancement of Science," by PROFESSOR G. A. MILLER;

"Western Meetings of Mathematicians," by PROFESSOR H. E. SLAUGHT;

"Summer Meeting of the American Mathematical Society," by PROFESSOR H. E. SLAUGHT;

"Notes and News" of events pertaining to mathematics, under the direction of a committee of which PROFESSOR FLORIAN CAJORI is chairman;

"The Napier Tercentenary Celebration," by PROFESSOR FLORIAN CAJORI, Colorado College;

"The Paris Report on Calculus in the Secondary Schools," EDITORIAL;

"California Teachers of Mathematics," EDITORIAL;

"Book Reviews" and announcements of new books in Mathematics, under the direction of a committee of which PROFESSOR W. H. BUSSEY, University of Minnesota, is chairman.

Fifty-four books have thus far been reviewed, each by a selected expert in his field.

Topics Involving a Minimum of Technical Treatment, such as the following:

"Maximum Parcels under the New Parcel Post Law," by PROFESSOR W. H. BUSSEY;

"Precise Measurements with a Steel Tape," by PROFESSOR G. R. DEAN, Missouri School of Mines;

"A Direct Definition of Logarithmic Derivative," by PROFESSOR E. R. HEDRICK;

"A Simple Formula for the Angle Between Two Planes," by PROFESSOR E. V. HUNTINGTON, Harvard University;

"On the Solutions of Linear Equations having Small Determinants," by PROFESSOR F. R. MOULTON, University of Chicago;

"The Accuracy of Interpolation in a Five-Place Table of Logarithms of Sines," by PROFESSORS A. M. KENYON and G. JAMES, Purdue University;

"A Theorem about Isogonal Conjugates," by DAVID F. BARROW, Harvard University;

"The Significance of the Weierstrass Theorem," by PROFESSOR E. R. HEDRICK;

"On the Impossibility of Certain Diophantine Equations and Systems of Equations," by PROFESSOR R. D. CARMICHAEL, Indiana University;

"A Computation Formula in Probability," by E. C. MOLINA, New York City;

"Two Geometrical Applications of the Method of Least Squares," by PROFESSOR J. L. COOLIDGE, Harvard University;

"A Puzzle Generalized," by PROFESSOR R. P. BAKER, University of Iowa;

"On Certain Diophantine Equations having Multiple Parameter Solutions," PROFESSOR R. D. CARMICHAEL;

"A Geometrical Discussion of the Regular Inscribed Hexagon," by J. Q. MCNATT, Florence Colo., and S. A. JOFFE, New York City;

"A Theorem in Number Theory connected with the Binomial Formula," by Professor D. N. LEHMER;

"An Application of Partial Derivatives to the Ellipse," by PROFESSOR M. O. TRIPP, Muncie, Ind.;

"A Curious Convergent Series," by PROFESSOR A. J. KEMPNER, University of Illinois;

"Optical Interpretations in Higher Geodesy," by PROFESSOR W. H. ROEVER, Washington University;

"A Problem in Number Theory," by PROFESSOR G. A. OSBORNE, Massachusetts Institute of Technology;

"Perfect Magic Squares for 1914," by V. M. SPUNAR, Chicago, Ill., and PROFESSOR B. L. REMICK, Manhattan, Kan.;

"The Construction of Conics under given Conditions," by Dr. B. M. WOODS, University of California;

"A Simple Method of Constructing the Normals to a Parabola," by PROFESSOR S. G. BARTON, University of Pennsylvania;

"Some Properties of the Normals to a Parabola," by PROFESSOR S. G. BARTON;

"Apparent Size of a Cube," by PROFESSOR A. M. HARDING, University of Arkansas;

"Residues of Certain Sums of Powers of Integers," by PROFESSOR T. M. PUTNAM, University of California;

"Groups of Figures in Elementary Geometry," by PROFESSOR G. A. MILLER, University of Illinois;

"On the Use of Partial Derivatives in Plotting Equations from their Curves," by PROFESSOR A. M. KENYON, Purdue University;

"A Method of Solving Numerical Equations," by S. A. COREY, Hiteman, Iowa;

"Sur un Paradoxe Algébrique Apparent," par G. LORIA, Université de Gêne;

"The Theorem of Rotation in Elementary Mechanics," by PROFESSOR E. V. HUNTINGTON, Harvard University;

"Groups of Subtraction and Division with Respect to a Modulus," by PROFESSOR G. A. MILLER, University of Illinois;

"Questions and Discussions," under the direction of PROFESSOR U. G. MITCHELL, University of Kansas;

"Problems Proposed and Solved," under the direction of PROFESSORS B. F. FINKEL, Drury College, and R. P. BAKER, University of Iowa.

Topics Involving Somewhat More Technical Treatment, designed to stimulate mathematical activity on the part of ambitious students and teachers. Such articles have occupied only about one-sixth of the entire space; for example, such as the following:

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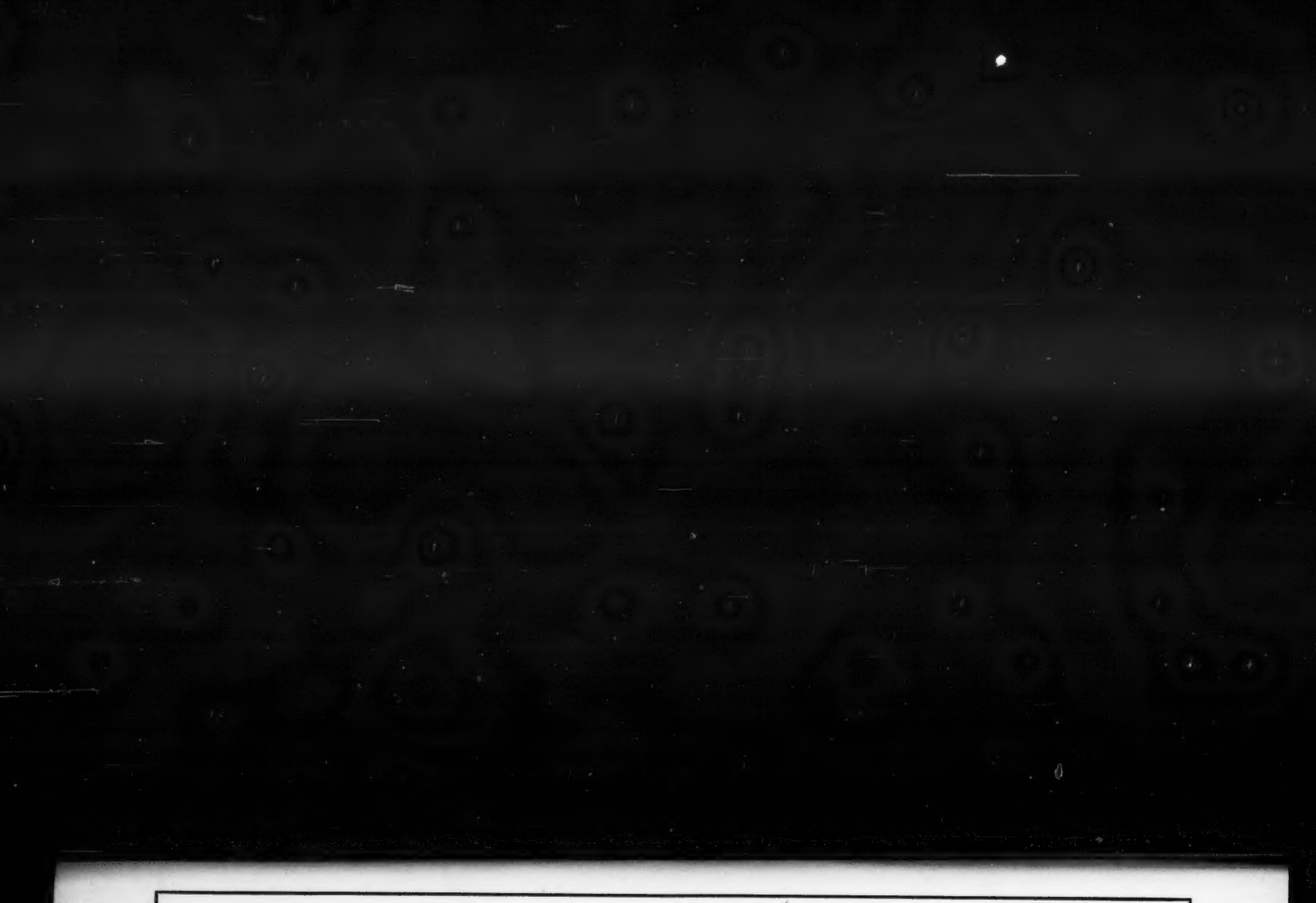

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